

On the optimal paving over MASAs in von Neumann algebras

BY SORIN POPA¹ AND STEFAAN VAES²

Abstract

We prove that if A is a singular MASA in a II_1 factor M and ω is a free ultrafilter, then for any $x \in M \ominus A$, with $\|x\| \leq 1$, and any $n \geq 2$, there exists a partition of 1 with projections $p_1, p_2, \dots, p_n \in A^\omega$ (i.e. a *paving*) such that $\|\sum_{i=1}^n p_i x p_i\| \leq 2\sqrt{n-1}/n$, and give examples where this is sharp. Some open problems on optimal pavings are discussed.

1 Introduction

A famous problem formulated by R.V. Kadison and I.M. Singer in 1959 asked whether the diagonal MASA (maximal abelian *-subalgebra) \mathcal{D} of the algebra $\mathcal{B}(\ell^2\mathbb{N})$, of all linear bounded operators on the Hilbert space $\ell^2\mathbb{N}$, satisfies the *paving property*, requiring that for any contraction $x = x^* \in \mathcal{B}(\ell^2\mathbb{N})$ with 0 on the diagonal, and any $\varepsilon > 0$, there exists a partition of 1 with projections $p_1, \dots, p_n \in \mathcal{D}$, such that $\|\sum_i p_i x p_i\| \leq \varepsilon$. This problem has been settled in the affirmative by A. Marcus, D. Spielman and N. Srivastava in [MSS13], with an actual estimate $n \leq 12^4 \varepsilon^{-4}$ for the *paving size*, i.e., for the minimal number $n = n(x, \varepsilon)$ of such projections.

In a recent paper [PV14], we considered a notion of paving for an arbitrary MASA in a von Neumann algebra $A \subset M$, that we called *so-paving*, which requires that for any $x = x^* \in M$ and any $\varepsilon > 0$, there exist $n \geq 1$, a net of partitions of 1 with n projections $p_{1,i}, \dots, p_{n,i} \in A$ and projections $q_i \in M$ such that $\|q_i(\sum_{k=1}^n p_{k,i} x p_{k,i} - a_i)q_i\| \leq \varepsilon$, $\forall i$, and $q_i \rightarrow 1$ in the *so-topology*.

This property is in general weaker than the classic Kadison-Singer norm paving, but it coincides with it for the diagonal MASA $\mathcal{D} \subset \mathcal{B}(\ell^2\mathbb{N})$. We conjectured in [PV14] that any MASA $A \subset M$ satisfies *so-paving*. We used the results in [MSS13] to check this conjecture for all MASAs in type I von Neumann algebras, and all Cartan MASAs in amenable von Neumann algebras and in group measure space factors arising from profinite actions, with the estimate $12^4 \varepsilon^{-4}$ for the *so-paving size* derived from [MSS13] as well.

We also showed in [PV14] that if A is the range of a normal conditional expectation, $E : M \rightarrow A$, and ω is a free ultrafilter on \mathbb{N} , then *so-paving* for $A \subset M$ is equivalent to the usual Kadison-Singer paving for the ultrapower MASA $A^\omega \subset M^\omega$, with the norm paving size for $A^\omega \subset M^\omega$ coinciding with the *so-paving size* for $A \subset M$. In the case A is a singular MASA in a II_1 factor M , norm-paving for the ultrapower inclusion $A^\omega \subset M^\omega$ has been established in [P13], with paving size $1250\varepsilon^{-3}$. This estimate was improved to $< 16\varepsilon^{-2} + 1$ in [PV14], while also shown to be $\geq \varepsilon^{-2}$ for arbitrary MASAs in II_1 factors.

In this paper we prove that the paving size for singular MASAs in II_1 factors is in fact $< 4\varepsilon^{-2} + 1$, and that for certain singular MASAs this is sharp. More precisely, we prove that for any contraction $x \in M^\omega$ with 0 expectation onto A^ω , and for any $n \geq 2$, there exists a partition of 1 with n projections $p_i \in A^\omega$ such that $\|\sum_{i=1}^n p_i x p_i\| \leq 2\sqrt{n-1}/n$. In fact, given any finite

¹Mathematics Department, UCLA, CA 90095-1555 (United States), popa@math.ucla.edu
Supported in part by NSF Grant DMS-1401718

²KU Leuven, Department of Mathematics, Leuven (Belgium), stefaan.vaes@wis.kuleuven.be
Supported by ERC Consolidator Grant 614195 from the European Research Council under the European Union's Seventh Framework Programme.

set of contractions $F \subset M^\omega \ominus A^\omega$, we can find a partition $p_1, \dots, p_n \in A^\omega$ that satisfies this estimate for all $x \in F$, so even the *multipaving size* for singular MASAs is $< 4\epsilon^{-2} + 1$.

To construct pavings satisfying this estimate, we first use Theorem 4.1(a) in [P13] to get a unitary $u \in A^\omega$ with $u^n = 1$, $\tau(u^k) = 0$, $1 \leq k \leq n-1$, such that any word with alternating letters from $\{u^k \mid 1 \leq k \leq n-1\}$ and $F \cup F^*$ has trace 0. This implies that for each $x \in F$ the set $X = \{u^{i-1}xu^{-i+1} \mid i = 1, 2, \dots, n\}$ satisfies the conditions $\tau(\Pi_{k=1}^m(x_{2k-1}x_{2k}^*)) = 0 = \tau(\Pi_{k=1}^m(x_{2k-1}^*x_{2k}))$, for all m and all $x_k \in X$ with $x_k \neq x_{k+1}$ for all k . We call *L-freeness* this property of a subset of a II_1 factor. We then prove the general result, of independent interest, that any L-free set of contractions $\{x_1, \dots, x_n\}$ satisfies the norm estimate $\|\sum_{i=1}^n x_i\| \leq 2\sqrt{n-1}$. We do this by first “dilating” $\{x_1, \dots, x_n\}$ to an L-free set of unitaries $\{U_1, \dots, U_n\}$ in a larger II_1 factor, for which we deduce the Kesten-type estimate $\|\sum_{i=1}^n U_i\| = 2\sqrt{n-1}$ from results in [AO74]. This implies the inequality for the L-free contractions as well. By applying this to $\{u^{i-1}xu^{1-i} \mid i = 1, \dots, n\}$ and taking into account that $\frac{1}{n}\sum_{i=1}^n u^{i-1}xu^{1-i} = \sum_{i=1}^n p_i x p_i$, where p_1, \dots, p_n are the minimal spectral projections of u , we get $\|\sum_{i=1}^n p_i x p_i\| \leq 2\sqrt{n-1}/n$, $\forall x \in F$.

We also notice that if M is a II_1 factor, $A \subset M$ is a MASA and $v \in M$ a self-adjoint unitary of trace 0 which is free with respect to A , then $\|\sum_{i=1}^n p_i v p_i\| \geq 2\sqrt{n-1}/n$ for any partition of 1 with projections in A^ω , with equality if and only if $\tau(p_i) = 1/n$, $\forall i$. A concrete example is when $M = L(\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z}))$, $A = L(\mathbb{Z})$ (which is a singular MASA in M by [P81]) and $v = v^* \in L(\mathbb{Z}/2\mathbb{Z}) \subset M$ denotes the canonical generator. This shows that the estimate $4\epsilon^{-2} + 1$ for the paving size is in this case optimal.

The constant $2\sqrt{n-1}$ is known to coincide with the spectral radius of the n -regular tree, and with the first eigenvalue less than n of n -regular Ramanujan graphs. Its occurrence in this context leads us to a more refined version of a conjecture formulated in [PV14], predicting that for any MASA $A \subset M$ which is range of a normal conditional expectation, any $n \geq 2$ and any contraction $x = x^* \in M$ with 0 expectation onto A , the infimum $\varepsilon(A \subset M; n, x)$ over all norms of pavings of x , $\|\sum_{i=1}^n p_i x p_i\|$, with n projections p_1, \dots, p_n in A^ω , $\sum p_i = 1$, is bounded above by $2\sqrt{n-1}/n$, and that in fact $\sup\{\varepsilon(A \subset M; n, x) \mid x = x^* \in M \ominus A, \|x\| \leq 1\} = 2\sqrt{n-1}/n$. Such an optimal estimate would be particularly interesting to establish for the diagonal MASA $\mathcal{D} \subset \mathcal{B}(\ell^2\mathbb{Z})$.

2 Preliminaries

A well known result of H. Kesten in [K58] shows that if \mathbb{F}_k denotes the free group with k generators h_1, \dots, h_k , and λ is the left regular representation of \mathbb{F}_k on $\ell^2\mathbb{F}_k$, then the norm of the *Laplacian operator* $L = \sum_{i=1}^k (\lambda(h_i) + \lambda(h_i^{-1}))$ is equal to $2\sqrt{2k-1}$. It was also shown in [K58] that, conversely, if k elements h_1, \dots, h_k in a group Γ satisfy $\|\sum_{i=1}^k \lambda(h_i) + \lambda(h_i^{-1})\| = 2\sqrt{2k-1}$, then h_1, \dots, h_k are freely independent, generating a copy of \mathbb{F}_k inside Γ . The calculation of the norm of L in [K58] uses the formalism of random walks on groups, but it really amounts to calculating the higher moments $\tau(L^{2n})$ and using the formula $\|L\| = \lim_m (\tau(L^{2m}))^{1/2m}$, where τ denotes the canonical (normal faithful) tracial state on the group von Neumann algebra $L(\mathbb{F}_k)$.

Kesten’s result implies that whenever u_1, \dots, u_k are freely independent Haar unitaries in a type II_1 factor M (i.e., u_1, \dots, u_k generate a copy of $L(\mathbb{F}_k)$ inside M), then one has $\|\sum_{i=1}^k u_i + u_i^*\| = 2\sqrt{2k-1}$. In particular, if M is the free group factor $L(\mathbb{F}_k)$ and $u_i = \lambda(h_i)$, where $h_1, \dots, h_k \in \mathbb{F}_k$ as above, then $\|\sum_{i=1}^k \alpha_i u_i + \overline{\alpha_i} u_i^*\| = 2\sqrt{2k-1}$, for any scalars $\alpha_i \in \mathbb{C}$ with $|\alpha_i| = 1$.

Estimates of norms of linear combinations of elements satisfying more general free independence

relations in group II_1 factors $L(\Gamma)$ have later been obtained in [L73], [B74], [AO74]³. These estimates involve elements in $L(\Gamma)$ (viewed as convolvers on $\ell^2\Gamma$) that are supported on a subset $\{g_1, \dots, g_n\} \subset \Gamma$ satisfying the following weaker freeness condition, introduced in [L73]: whenever $k \geq 1$ and $i_s \neq j_s, j_s \neq i_{s+1}$ for all s , we have that

$$g_{i_1} g_{j_1}^{-1} \cdots g_{i_k} g_{j_k}^{-1} \neq e.$$

In [B74] and [AO74], this is called the *Leinert property* and it is proved to be equivalent with $\{g_1^{-1} g_2, \dots, g_1^{-1} g_n\}$ freely generating a copy of \mathbb{F}_{n-1} . The most general calculation of norms of elements $x = \sum_i c_i \lambda(g_i) \in L(\Gamma)$, supported on a Leinert set $\{g_i\}_i$, with arbitrary coefficients $c_i \in \mathbb{C}$, was obtained by Akemann and Ostrand in [AO74]. The calculation shows in particular that if $\{g_1, \dots, g_n\}$ satisfies Leinert's freeness condition then $\|\sum_{i=1}^n \lambda(g_i)\| = 2\sqrt{n-1}$. Since $h_1, \dots, h_k \in \Gamma$ freely independent implies $\{h_i, h_i^{-1} \mid 1 \leq i \leq k\}$ is a Leinert set, the result in [AO74] does recover Kesten's theorem as well. Like in [K58], the norm of an element of the form $L = \sum_{i=1}^n c_i \lambda(g_i)$ in [AO74] is calculated by evaluating $\lim_n \tau((L^* L)^n)^{1/2n}$ (by computing the generating function of the moments of $L^* L$).

An argument similar to [K58] was used in [Le96] to prove that, conversely, if some elements g_1, \dots, g_n in a group Γ satisfy $\|\sum_{i=1}^n \lambda(g_i)\| = 2\sqrt{n-1}$, then g_1, \dots, g_n is a Leinert set. On the other hand, note that if g_1, \dots, g_n are n arbitrary elements in an arbitrary group Γ and we denote $L = \sum_{i=1}^n \lambda(g_i)$ the corresponding Laplacian, then the n 'th moment $\tau((L^* L)^n)$ is bounded from below by the n 'th moment of the Laplacian obtained by taking g_i to be the generators of \mathbb{F}_n . Thus, we always have $\|\sum_{i=1}^n \lambda(g_i)\| \geq 2\sqrt{n-1}$. More generally, if v_1, \dots, v_n are unitaries in a von Neumann algebra M with normal faithful trace state τ , such that any word $v_{i_1} v_{j_1}^* v_{i_2} v_{j_2}^* \dots v_{i_m} v_{j_m}^*$, $\forall m \geq 1, \forall 1 \leq i_k, j_k \leq n$, has trace with non-negative real part, then $\|\sum_{i=1}^n v_i\| \geq 2\sqrt{n-1}$. In particular, for any unitaries $u_1, \dots, u_n \in M$ one has $\|\sum_{i=1}^n u_i \otimes \overline{u_i}\| \geq 2\sqrt{n-1}$.

For convenience, we state below some norm calculations from [AO74], formulated in the form that will be used in the sequel:

Proposition 2.1 ([AO74]). *If $v_1, v_2, \dots, v_{n-1} \in M$ are freely independent Haar unitaries, then*

$$\|1 + \sum_{i=1}^{n-1} v_i\| = 2\sqrt{n-1}. \quad (2.1)$$

Also, if $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{C}$, $\sum_i |\alpha_i|^2 = 1$, then

$$\|\alpha_0 1 + \sum_{i=1}^{n-1} \alpha_i v_i\| \leq 2\sqrt{1-1/n}. \quad (2.2)$$

Note that (2.1) above shows in particular that if $p, q \in M$ are projections with $\tau(p) = 1/2$ and $\tau(q) = 1/n$, for some $n \geq 3$, and they are freely independent, then $\|qpq\| = 1/2 + \sqrt{n-1}/n$. Indeed, any two such projections can be thought of as embedded into $L(\mathbb{F}_2)$ with p and q lying in the MASAs of the two generators, $p \in A_1$, respectively $q \in A_2$. Denote $v = 2p - 1$. Let $q_1 = q, q_2, \dots, q_n \in A_2$ be mutually orthogonal projections of trace $1/n$ and denote $u = \sum_{j=1}^n \lambda^{j-1} q_j$, where $\lambda = 2\exp(2\pi i/n)$. It is then easy to see that the elements $v_k = v u^k v u^{-k}$, $k = 1, 2, \dots, n-1$ are freely independent Haar unitaries. By (2.1) we thus have $\|\sum_{k=0}^{n-1} u^k v u^{-k}\| = \|1 + \sum_{k=1}^{n-1} v u^k v u^{-k}\| = 2\sqrt{n-1}$. But $\sum_{k=0}^{n-1} u^k v u^{-k} = n(\sum_{j=1}^n q_j v q_j)$, implying that

³See also the more "rough" norm estimates for elements in $L(\mathbb{F}_n)$ obtained by R. Powers in 1967 in relation to another problem of Kadison, but published several years later in [Po75], and which motivated in part the work in [AO74].

$$\|qvq\| = \|q(2p-1)q\| = 2\sqrt{n-1}/n = 2\sqrt{\tau(q)(1-\tau(q))}$$

or equivalently

$$\|qpq\| = 1/2 + \sqrt{n-1}/n = \tau(p) + \sqrt{\tau(q)(1-\tau(q))}.$$

The computation of the norm of the product of freely independent projections q, p of arbitrary trace in M (in fact, of the whole spectral distribution of qpq) was obtained by Voiculescu in [Vo86], as one of the first applications of his multiplicative free convolution (which later became a powerful tool in free probability). We recall here these norm estimates, which in particular show that the first of the above norm calculations holds true for projections q of arbitrary trace (see also [ABH87] for the case $\tau(q) = 1/n$, $\tau(p) = 1/m$, for integers $n \geq m \geq 2$):

Proposition 2.2 ([Vo86]). *If $p, q \in M$ are freely independent projections with $\tau(q) \leq \tau(p) \leq 1/2$, then*

$$\|qpq\| = \tau(p) + \tau(q) - 2\tau(p)\tau(q) + 2\sqrt{\tau(p)\tau(1-p)\tau(q)\tau(1-q)}. \quad (2.3)$$

If in addition $\tau(p) = 1/2$ and we denote $v = 2p - 1$, then

$$\|qvq\| = 2\sqrt{\tau(q)\tau(1-q)}. \quad (2.4)$$

3 L-free sets of contractions and their dilation

Recall from [P13] that two selfadjoint sets $X, Y \subset M \ominus \mathbb{C}1$ of a tracial von Neumann algebra M are called *freely independent sets*⁴ if the trace of any word with letters alternating from X and Y is equal to 0. Also, a subalgebra $B \subset M$ is called *freely independent of a set X* , if X and $B \ominus \mathbb{C}1$ are freely independent as sets. Several results were obtained in [P13] about constructing a “large subalgebra” B inside a given subalgebra $Q \subset M$ that is freely independent of a given countable set X . Motivated by a condition appearing in one such result, namely [P13, Theorem 4.1], and by a terminology used in [AO74], we consider in this paper the following free independence condition for arbitrary elements in tracial algebras:

Definition 3.1. Let (M, τ) be a von Neumann algebra with a normal faithful tracial state. A subset $X \subset M$ is called *L-free*⁵ if

$$\tau(x_1 x_2^* \cdots x_{2k-1} x_{2k}^*) = 0 \quad \text{and} \quad \tau(x_1^* x_2 \cdots x_{2k-1}^* x_{2k}) = 0,$$

whenever $k \geq 1$, $x_1, \dots, x_{2k} \in X$ and $x_i \neq x_{i+1}$ for all $i = 1, \dots, 2k-1$.

Note that if the subset X in the above definition is taken to be contained in the set of canonical unitaries $\{u_g \mid g \in \Gamma\}$ of a group von Neumann algebra $M = L(\Gamma)$, i.e. $X = \{u_g \mid g \in F\}$ for some subset $F \subset \Gamma$, then L-freeness of X amounts to F being a Leinert set. But the key example of an L-free set that is important for us here occurs from a diffuse algebra B that is free independent from a set $Y = Y^* \subset M \ominus \mathbb{C}1$: given any $y_1, \dots, y_n \in Y$ and any unitary element $u \in \mathcal{U}(B)$ with $\tau(u^k) = 0$, $1 \leq k \leq n-1$, the set $\{u^{k-1} y_k u^{-k+1} \mid 1 \leq k \leq n\}$ is L-free.

⁴We specifically consider this condition for subsets $X, Y \subset M \ominus \mathbb{C}1$, not to be confused with the freeness of the von Neumann algebras generated by X and Y .

⁵Note that this notion is not the same as (and should not be confused with) the notion of L-sets used in [Pi92].

Note that we do need to impose both conditions on the traces being zero in Definition 3.1, because we cannot deduce $\tau(x_1^*x_2x_3^*x_1) = 0$ from $\tau(y_1y_2^*y_3y_4^*) = 0$ for all $y_i \in X$ with $y_1 \neq y_2$, $y_2 \neq y_3$, $y_3 \neq y_4$. However, if $X \subset \mathcal{U}(M)$ consists of unitaries, then only one set of conditions is sufficient. We in fact have:

Lemma 3.2. *Let $X = \{u_1, \dots, u_n\} \subset \mathcal{U}(M)$. Then the following conditions are equivalent*

- (a) X is an L -free set.
- (b) $\tau(u_{i_1}u_{j_1}^* \cdots u_{i_k}u_{j_k}^*) = 0$ whenever $k \geq 1$ and $i_s \neq j_s$, $j_s \neq i_{s+1}$ for all s .
- (c) $u_1^*u_2, \dots, u_1^*u_n$ are free generators of a copy of $L(\mathbb{F}_{n-1})$.

Proof. This is a trivial verification. \square

Corollary 3.3. *If $\{u_1, \dots, u_n\}$ is an L -free set of unitaries in $\mathcal{U}(M)$, then $\|\sum_{i=1}^n u_i\| = 2\sqrt{n-1}$. Moreover, if $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ with $\sum_{i=1}^n |\alpha_i|^2 \leq 1$, then*

$$\left\| \sum_{i=1}^n \alpha_i u_i \right\| \leq 2\sqrt{1-1/n}.$$

Proof. Since $\|\sum_{i=1}^n \alpha_i u_i\| = \|\alpha_1 1 + \sum_{i=2}^n \alpha_i u_1^* u_i\|$, the statement follows by applying (2.2) to the freely independent Haar unitaries $v_j = u_1^* u_j$, $2 \leq j \leq n$. \square

Proposition 3.4. *Let M be a finite von Neumann algebra with a faithful tracial state τ . If $\{x_1, \dots, x_n\} \subset M$ is an L -free set with $\|x_i\| \leq 1$ for all i , then there exists a tracial von Neumann algebra (\mathcal{M}, τ) , a trace preserving unital embedding $M \subset \mathcal{M}$ and an L -free set of unitaries $\{U_1, \dots, U_n\} \subset \mathcal{U}(\widetilde{\mathcal{M}})$ with $\widetilde{\mathcal{M}} = M_{n+1}(\mathbb{C}) \otimes \mathcal{M}$ so that, denoting by $(e_{ij})_{i,j=0,\dots,n}$ the matrix units of $M_{n+1}(\mathbb{C})$, we have $e_{00}U_i e_{00} = x_i$ for all i .*

Proof. Define $\mathcal{M} = M * L(\mathbb{F}_{n(n-1)})$ and denote by $u_{i,j}$, $i \neq j$, free generators of $L(\mathbb{F}_{n(n-1)})$. For every $i \in \{1, \dots, n\}$, define

$$c_i = \sqrt{1 - x_i x_i^*} \quad \text{and} \quad d_i = -\sqrt{1 - x_i^* x_i}.$$

Put $\widetilde{\mathcal{M}} = M_{n+1}(\mathbb{C}) \otimes \mathcal{M}$ and define the unitary elements $U_i \in \mathcal{U}(\widetilde{\mathcal{M}})$ given by

$$U_i = (e_{00} \otimes x_i) + (e_{ii} \otimes x_i^*) + (e_{0i} \otimes c_i) + (e_{i0} \otimes d_i) + \sum_{j \neq i} (e_{jj} \otimes u_{i,j}).$$

Note that U_i is the direct sum of the unitary

$$\begin{pmatrix} x_i & c_i \\ d_i & x_i^* \end{pmatrix} \text{ in positions } 0 \text{ and } i, \text{ and the unitary } \bigoplus_{j \neq i} u_{i,j} \text{ in the positions } j \neq i.$$

By construction, we have that $e_{00}U_i e_{00} = x_i e_{00}$. So, it remains to prove that $\{U_1, \dots, U_n\}$ is L -free.

Take $k \geq 1$ and indices i_s, j_s such that $i_s \neq j_s$, $j_s \neq i_{s+1}$ for all s . We must prove that

$$\tau(U_{i_1}U_{j_1}^* \cdots U_{i_k}U_{j_k}^*) = 0. \quad (3.1)$$

Consider $V := U_{i_1}U_{j_1}^* \cdots U_{i_k}U_{j_k}^*$ as a matrix with entries in \mathcal{M} . Every entry of this matrix is a sum of “words” with letters

$$\{x_i, x_i^*, c_i, d_i \mid i = 1, \dots, n\} \cup \{u_{i,j}, u_{i,j}^* \mid i \neq j\}.$$

We prove that every word that appears in a diagonal entry V_{ii} of V has zero trace. The following types of words appear.

1° Words without any of the letters $u_{a,b}$ or $u_{a,b}^*$. These words only appear as follows:

- in the entry V_{00} as $x_{i_1}x_{j_1}^* \cdots x_{i_k}x_{j_k}^*$, which has zero trace;
- if $i_1 = j_k = i$, in the entry V_{ii} as $w = d_i x_{j_1}^* x_{i_2} x_{j_2}^* \cdots x_{i_{k-1}} x_{j_{k-1}}^* x_{i_k} d_i^*$. Then we have

$$\begin{aligned}\tau(w) &= \tau(x_{j_1}^* x_{i_2} \cdots x_{j_{k-1}}^* x_{i_k} d_i^* d_i) \\ &= \tau(x_{j_1}^* x_{i_2} \cdots x_{j_{k-1}}^* x_{i_k}) - \tau(x_{j_1}^* x_{i_2} \cdots x_{j_{k-1}}^* x_{i_k} x_i^* x_i) \\ &= 0 - \tau(x_{i_1} x_{j_1}^* \cdots x_{i_k} x_{j_k}^*) = 0 ,\end{aligned}$$

because $i = i_1$ and $i = j_k$.

2° Words with exactly one letter of the type $u_{a,b}$ or $u_{a,b}^*$. These words have zero trace because $\tau(Mu_{a,b}M) = \{0\}$.

3° Words w with two or more letters of the type $u_{a,b}$ or $u_{a,b}^*$. Consider two consecutive such letters in w , i.e. a subword of w of the form

$$u_{i,j}^\varepsilon w_0 u_{i',j'}^{\varepsilon'}$$

with $\varepsilon, \varepsilon' = \pm 1$ and where w_0 is a word with letters from $\{x_i, x_i^*, c_i, d_i \mid i = 1, \dots, n\}$. We distinguish three cases.

- $(\varepsilon', i', j') \neq (-\varepsilon, i, j)$.
- $u_{i,j}^\varepsilon w_0 u_{i,j}^*$.
- $u_{i,j}^* w_0 u_{i,j}^\varepsilon$.

To prove that $\tau(w) = 0$, it suffices to prove that in the last two cases, we have that $\tau(w_0) = 0$.

A subword of the form $u_{i,j}^\varepsilon w_0 u_{i,j}^*$ can only arise from the jj -entry of

$$U_{i_s} U_{j_s}^* \cdots U_{i_t} U_{j_t}^* \quad \text{with } i_s = j_t = i, j_s = i_t = j$$

(and thus, $t \geq s + 2$). In that case,

$$w_0 = c_j^* x_{i_{s+1}} x_{j_{s+1}}^* \cdots x_{i_{t-1}} x_{j_{t-1}}^* c_j .$$

Thus,

$$\begin{aligned}\tau(w_0) &= \tau(x_{i_{s+1}} x_{j_{s+1}}^* \cdots x_{i_{t-1}} x_{j_{t-1}}^* c_j c_j^*) \\ &= \tau(x_{i_{s+1}} x_{j_{s+1}}^* \cdots x_{i_{t-1}} x_{j_{t-1}}^*) - \tau(x_{i_{s+1}} x_{j_{s+1}}^* \cdots x_{i_{t-1}} x_{j_{t-1}}^* x_j x_j^*) \\ &= 0 - \tau(x_{j_s} x_{i_{s+1}}^* \cdots x_{j_{t-1}}^* x_{i_t}) = 0 ,\end{aligned}$$

because $j = j_s$ and $j = i_t$.

Finally, a subword of the form $u_{i,j}^* w_0 u_{i,j}^\varepsilon$ can only arise from the jj -entry of

$$U_{j_{s-1}}^* U_{i_s} \cdots U_{j_{t-1}}^* U_{i_t} \quad \text{with } j_{s-1} = i_t = i, i_s = j_{t-1} = j$$

(and thus, $t \geq s + 2$). In that case,

$$w_0 = d_j x_{j_s}^* x_{i_{s+1}} \cdots x_{j_{t-2}}^* x_{i_{t-1}} d_j^* .$$

As above, it follows that $\tau(w_0) = 0$.

So, we have proved that every word that appears in a diagonal entry V_{ii} of V has trace zero. Then also $\tau(V) = 0$ and it follows that $\{U_1, \dots, U_n\}$ is an L-free set of unitaries. \square

Corollary 3.5. *Let (M, τ) be a finite von Neumann algebra with a faithful normal tracial state. If $\{x_1, \dots, x_n\} \subset M$ is L-free with $\|x_i\| \leq 1$ for all i , then*

$$\left\| \sum_{i=1}^n x_i \right\| \leq 2\sqrt{n-1}.$$

More generally, given any complex scalars $\alpha_1, \dots, \alpha_n$ with $\sum_{i=1}^n |\alpha_i|^2 \leq 1$, we have

$$\left\| \sum_{i=1}^n \alpha_i x_i \right\| \leq 2\sqrt{1-1/n}.$$

Proof. Assuming $n \geq 2$, with the notations from Proposition 3.4 and by using Corollary 3.3, we have $\left\| \sum_{i=1}^n \alpha_i U_i \right\| \leq 2\sqrt{1-1/n}$. Reducing with the projection e_{00} , it follows that

$$\left\| \sum_{i=1}^n \alpha_i x_i \right\| \leq 2\sqrt{1-1/n}.$$

\square

4 Applications to paving problems

Like in [P13], [PV14], if $\mathcal{A} \subset \mathcal{M}$ is a MASA in a von Neumann algebra and $x \in \mathcal{M}$, then we denote by $n(\mathcal{A} \subset \mathcal{M}; x, \varepsilon)$ the smallest n for which there exist projections $p_1, \dots, p_n \in \mathcal{A}$ and $a \in \mathcal{A}$ such that $\|a\| \leq \|x\|$, $\sum_{i=1}^n p_i = 1$ and $\left\| \sum_{i=1}^n p_i x p_i - a \right\| \leq \varepsilon \|x\|$ (with the convention that $n(\mathcal{A} \subset \mathcal{M}; x, \varepsilon) = \infty$ if no such finite partition exists), and call it the *paving size* of x .

Recall also from [D54] that a MASA \mathcal{A} in a von Neumann algebra \mathcal{M} is called *singular*, if the only unitary elements in \mathcal{M} that normalize \mathcal{A} are the unitaries in \mathcal{A} .

Theorem 4.1. *Let $A_n \subset M_n$ be a sequence of singular MASAs in finite von Neumann algebras and ω a free ultrafilter on \mathbb{N} . Denote $\mathbf{M} = \prod_{\omega} M_n$ and $\mathbf{A} = \prod_{\omega} A_n$. Given any countable set of contractions $X \subset \mathbf{M} \ominus \mathbf{A}$ and any integer $n \geq 2$, there exists a partition of 1 with projections $p_1, \dots, p_n \in \mathbf{A}$ such that*

$$\left\| \sum_{j=1}^n p_j x p_j \right\| \leq 2\sqrt{n-1}/n, \quad \text{for all } x \in X.$$

In particular, the paving size of $\mathbf{A} \subset \mathbf{M}$,

$$n(\mathbf{A} \subset \mathbf{M}; \varepsilon) \stackrel{\text{def}}{=} \sup\{n(\mathbf{A} \subset \mathbf{M}; x, \varepsilon) \mid x = x^* \in \mathbf{M} \ominus \mathbf{A}\},$$

is less than $4\varepsilon^{-2} + 1$, for any $\varepsilon > 0$.

Proof. By Theorem 4.1(a) in [P13], there exists a diffuse abelian von Neumann subalgebra $A_0 \subset \mathbf{A}$ such that for any $k \geq 1$, any word with alternating letters $x = x_0 \prod_{i=1}^k (v_i x_i)$ with $x_i \in X$, $1 \leq i \leq k-1$, $x_0, x_k \in X \cup \{1\}$, $v_i \in A_0 \ominus \mathbb{C}1$, has trace equal to 0.

This implies that if $p_1, \dots, p_n \in \mathbf{A}$ are projections of trace $1/n$ summing up to 1 and we denote $u = \sum_{j=1}^n \lambda^{j-1} p_j$, where $\lambda = \exp(2\pi i/n)$, then for any $x \in X$ the set $\{u^{i-1} x u^{-i+1} \mid i = 1, 2, \dots, n\}$ is L-free. Since $\frac{1}{n} \sum_{i=1}^n u^{i-1} x u^{-i+1} = \sum_{i=1}^n p_i x p_i$, where p_1, \dots, p_n are the minimal spectral projections of u , by Proposition 3.4 it follows that for all $x \in X$ we have

$$\|\sum_{i=1}^n p_i x p_i\| = \frac{1}{n} \|\sum_{i=1}^n u^{i-1} x u^{-i+1}\| \leq 2\sqrt{n-1}/n.$$

To derive the last part, let $\varepsilon > 0$ and denote by n the integer with the property that $2n^{-1/2} \leq \varepsilon < 2(n-1)^{-1/2}$. If $x \in \mathbf{M} \ominus \mathbf{A}$, $\|x\| \leq 1$, and $p_1, \dots, p_n \in \mathbf{A}$ are mutually orthogonal projections of trace $1/n$ that satisfy the free independence relation with $X = \{x\}$ as above, then $n < 4\varepsilon^{-2} + 1$ and we have

$$\|\sum_{i=1}^n p_i x p_i\| \leq 2\sqrt{n-1}/n \leq \varepsilon,$$

showing that $n(\mathbf{A} \subset \mathbf{M}; x, \varepsilon) < 4\varepsilon^{-2} + 1$. \square

Remark 4.2. The above result suggests that an alternative way of measuring the *so*-paving size over a MASA in a von Neumann algebra $A \subset M$ admitting a normal conditional expectation, is by considering the quantity

$$\varepsilon(A \subset M; n) \stackrel{\text{def}}{=} \sup_{x \in (M_h^\omega \ominus A^\omega)_1} (\inf \{ \|\sum_{i=1}^n p_i x p_i\| \mid p_i \in \mathcal{P}(A^\omega), \Sigma_i p_i = 1 \}).$$

With this notation, the above theorem shows that for a singular MASA in a II_1 factor $A \subset M$, one has $\varepsilon(A \subset M; n) \leq 2\sqrt{n-1}/n$, $\forall n \geq 2$, a formulation that's slightly more precise than the estimate $n_s(A \subset M; \varepsilon) = n(A^\omega \subset M^\omega; \varepsilon) < 4\varepsilon^{-2} + 1$. Also, the conjecture (2.8.2° in [PV14]) about the *so*-paving size can this way be made more precise, by asking whether $\varepsilon(A \subset M; n) \leq 2\sqrt{n-1}/n$, $\forall n$, for any MASA with a normal conditional expectation $A \subset M$. It seems particularly interesting to study this question in the classical Kadison-Singer case of the diagonal MASA $\mathcal{D} \subset \mathcal{B} = \mathcal{B}(\ell^2 \mathbb{N})$, and more generally for Cartan MASAs $A \subset M$. So far, the solution to the Kadison-Singer paving problem in [MSS13] shows that $\varepsilon(\mathcal{D} \subset \mathcal{B}; n) \leq 12n^{-1/4}$.

Also, while by [CEKP07] one has $n(\mathcal{D} \subset \mathcal{B}; \varepsilon) \geq \varepsilon^{-2}$ and by [PV14] one has $n_s(A \subset M; \varepsilon) = n(A^\omega \subset M^\omega; \varepsilon) \geq \varepsilon^{-2}$, for any MASA in a II_1 factor $A \subset M$, it would be interesting to decide whether $\varepsilon(\mathcal{D} \subset \mathcal{B}; n)$ and $\varepsilon(A \subset M; n)$ are in fact bounded from below by $2\sqrt{n-1}/n$, $\forall n$.

For a singular MASA in a II_1 factor, $A \subset M$, combining 4.1 with such a lower bound would show that $\varepsilon(A \subset M; n) = 2\sqrt{n-1}/n$, $\forall n$. While we could not prove this general fact, let us note here that for certain singular MASAs this equality holds indeed.

Proposition 4.3. 1° *Let M be a II_1 factor and $A \subset M$ a MASA. Assume $v \in M$ is a unitary element with $\tau(v) = 0$ such that A is freely independent of the set $\{v, v^*\}$ (i.e., any alternating word in $A \ominus \mathbb{C}1$ and $\{v, v^*\}$ has trace 0). Then for any partition of 1 with projections $p_1, \dots, p_n \in A^\omega$ we have $\|\sum_{i=1}^n p_i v p_i\| \geq 2\sqrt{n-1}/n$, with equality iff all p_i have trace $1/n$. Also, $\varepsilon(A \subset M; n) \geq 2\sqrt{n-1}/n$, $\forall n$.*

2° *If $M = L(\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z}))$, $A = L(\mathbb{Z})$ and $v = v^*$ denotes the canonical generator of $L(\mathbb{Z}/2\mathbb{Z})$, then $\varepsilon(A \subset M; v, n) = \varepsilon(A \subset M; n) = 2\sqrt{n-1}/n$, $\forall n$.*

Proof. The free independence assumption in 1° implies that $A^\omega \ominus \mathbb{C}$ and $\{v, v^*\}$ are freely independent sets as well. This in turn implies that for each i , the projections p_i and $v p_i v^*$ are freely independent, and so by Proposition 2.2 one has $\|p_i v p_i\| = \|p_i v p_i v^*\| = 2\sqrt{\tau(p_i)(1 - \tau(p_i))}$.

Thus, if one of the projections p_i has trace $\tau(p_i) > 1/n$, then $\|\Sigma_j p_j v p_j\| \geq \|p_i v p_i\| > 2\sqrt{n-1}/n$, while if $\tau(p_i) = 1/n$, $\forall i$, then $\|\Sigma_j p_j v p_j\| = 2\sqrt{n-1}/n$.

By applying 1° to part 2°, then using 4.1 and the fact that $A = L(\mathbb{Z})$ is singular in $M = L(\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z}))$ (cf. [P81]), proves the last part of the statement. \square

References

- [AO74] C.A. Akemann and P.A. Ostrand, Computing norms in group C*-algebras. *Amer. J. Math.* **98** (1976), 1015-1047.
- [ABH87] J. Anderson, B. Blackadar and U. Haagerup, Minimal projections in the reduced group C*-algebra of $\mathbb{Z}_n * \mathbb{Z}_m$, *J. Operator Theory* **26** (1991), 3-23.
- [B74] M. Bozejko, On $\Lambda(p)$ sets with minimal constant in discrete noncommutative groups. *Proc. Amer. Math. Soc.* **51** (1975), 407-412.
- [CEKP07] P. Casazza, D. Edidin, D. Kalra and V.I. Paulsen, Projections and the Kadison-Singer problem. *Oper. Matrices* **1** (2007), 391-408.
- [D54] J. Dixmier, Sous-anneaux abéliens maximaux dans les facteurs de type fini, *Ann. of Math.* **59** (1954), 279-286.
- [KS59] R.V. Kadison and I.M. Singer, Extensions of pure states, *Amer. J. Math.* **81** (1959), 383-400.
- [K58] H. Kesten, Symmetric random walks on groups. *Trans. Amer. Math. Soc.* **92** (1959), 336-354.
- [Le96] F. Lehner, A characterization of the Leinert property. *Proc. Amer. Math. Soc.* **125** (1997), 3423-3431.
- [L73] M. Leinert, Faltungsoperatoren auf gewissen diskreten Gruppen. *Studia Math.* **52** (1974), 149-158.
- [MSS13] A.W. Marcus, D.A. Spielman and N. Srivastava, Interlacing families II: mixed characteristic polynomials and the Kadison-Singer problem. *Ann. of Math.* **182** (2015), 327-350.
- [Pi92] G. Pisier, Multipliers and lacunary sets in non-amenable groups, *American J. Math.* **117** (1995), 337-376.
- [P81] S. Popa, Orthogonal pairs of *-subalgebras in finite von Neumann algebras, *J. Operator Theory*, **9** (1983), 253-268.
- [P13] S. Popa, A II_1 factor approach to the Kadison-Singer problem. *Comm. Math. Phys.* **332** (2014), 379-414.
- [PV14] S. Popa and S. Vaes, Paving over arbitrary MASAs in von Neumann algebra, to appear in *Analysis and PDE*. [arXiv:1412.0631](https://arxiv.org/abs/1412.0631)
- [Po75] R. Powers: Simplicity of the C*-algebra associated with the free group on two generators, *Duke Mathematical Journal* **42** (1975), 151-156.
- [Vo86] D. Voiculescu, Multiplication of certain noncommuting random variables. *J. Operator Theory* **18** (1987), 223-235.